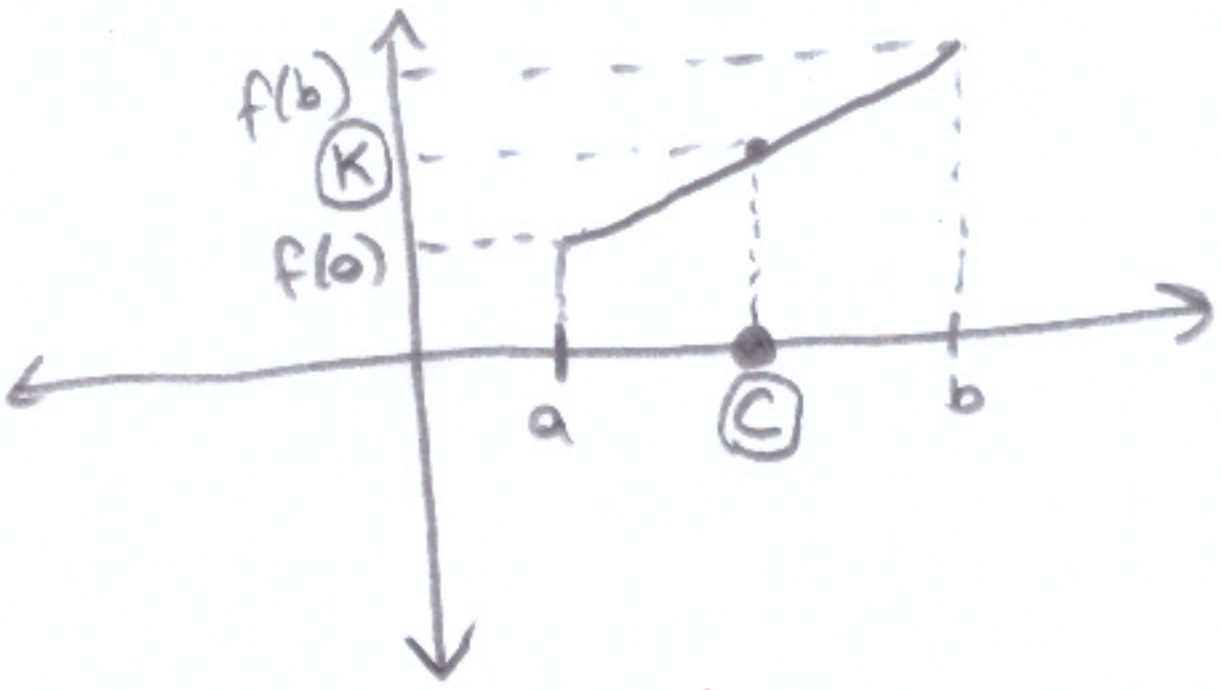
MAT 3012 - MIDTERM 1.DERS

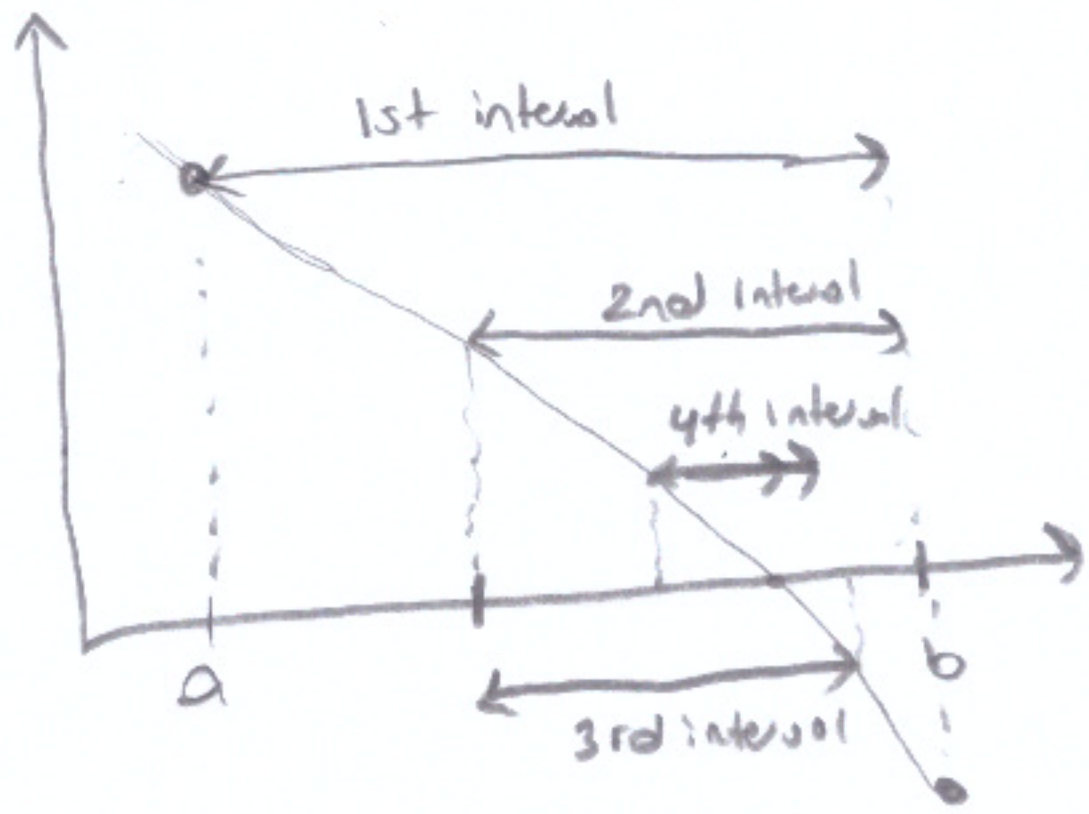
Intermediate Value Theorem

If $f(x)$ is continuous function within the interval $a \le x \le b$ and K is the number between $f(a)$ and $f(b)$. Then there exist a number $a \le c \le b$ for which $f(c) = K$



Bisection Method

We look for an approximation to p such that $f(p)=0$ in the interval $[a,b]$. Subsequent approximation should differ by less than some tolerance. This will only work if we select $[a,b]$ so that $f(a).f(b) < 0$



Note that it also generates a sequence $p_1, p_2 \ldots$

$$p_n = \frac{a_n + b_n}{2}$$

Stopping criteria

- ★ $|b_n - a_n| < tol$
- ★ $|p_n - p_{n-1}| < tol$
- ★ $f(p_n) < tol$
- ★ $\dfrac{|p_n - p_{n-1}|}{p_n} < \varepsilon$

After n, iteration Error $= \dfrac{b-a}{2^n}$

VARYANS eğitim

Examples

1-) Find the root of function $f(x)=x^3-x-1$ at $a=0$ $b=2$ with $Tol=0,01$

Iteration	a	b	p	f(a)	f(b)	f(p)
1	0	2	1	-1	5	-1
2	1	2	1.5	-1	5	0.875
3	1	1.5	1.25	-1	0.875	-0.297
4	1.25	1.5	1.375	-0,297	0.875	0.225
5	1.25	1.375	1.313	-0.297	0.225	-0.052
6	1.313	1.375	1.344	-0.052	0.225	0.083
7	1.313	1.344	1.328	-0.052	0.083	0.015
8	1.313	1.328	1.320	-0.052	0.015	-0.019
9	1.320	1.328	1.324	-0.019	0.015	-0.002

$f(b).f(p)<0 \quad a=p$

2-) Use bisection method to find root the function $f(x)=3x+\sin x-e^x$ $a=0$ and $b=1$

The exact solution $p=0,36042170296\ldots$

step	a	b	p	f(a)	f(b)	f(p)
1	0	1	0,5	-1	1.123	0,331
2	0	0,5	0,25	-1	0,331	-0,287
3	0,25	0,5	0,375	-0,287	0,331	0,036
4	0,25	0,375	0,3125	-0,287	0,036	-0,122
5	0,3125	0,375	0,34375	-0,122	0,036	-0,042

$p_5=0,34375$

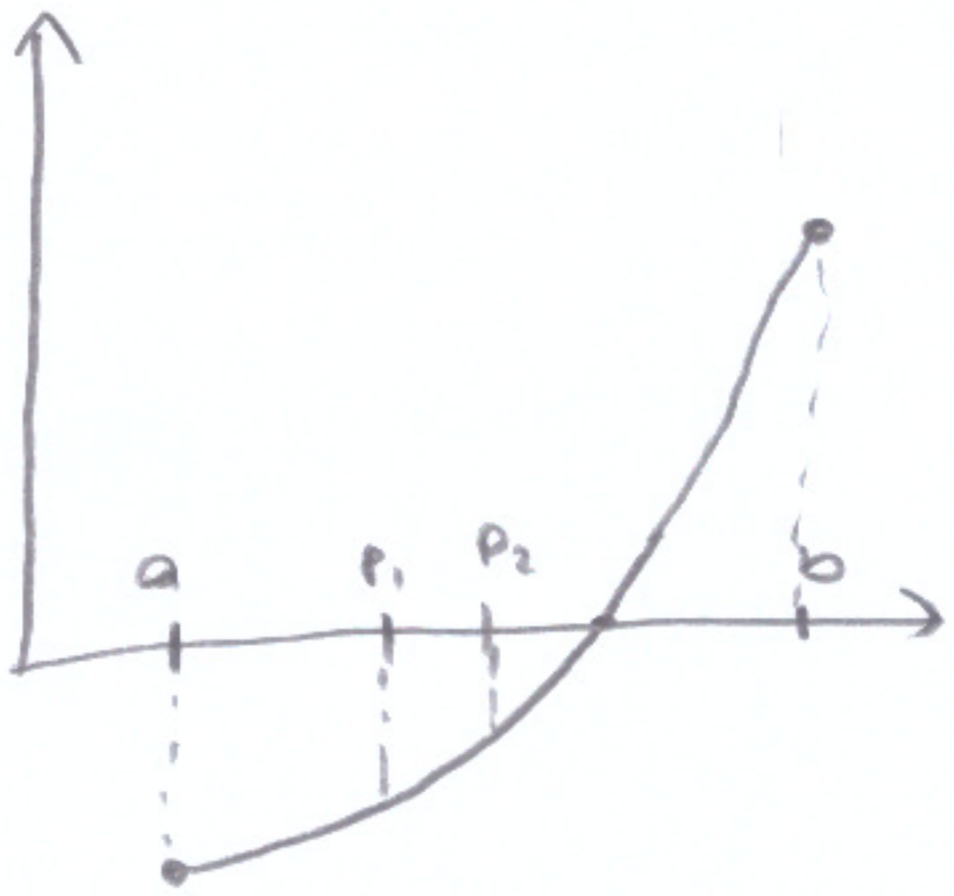
$|p_5-p|=0,01667$

$Error=\frac{1-0}{2^5}=\frac{1}{32}=0,03125$

3-) Use bisection method to find p_3 for $f(x)=\sqrt{x}-\cos x$ on $[0,1]$

step	a	b	p	f(a)	f(b)	f(p)
1	0	1	0,5	-1	0,46	-0,017
2	0,5	1	0,75	-0,17	0,46	0,134
3	0,5	0,75	0,625	-0,17	0,134	-0,02

False Position Method



Algorithms

Set $a_1 = a$ and $b_1 = b$

$$P_1 = \frac{a_1 . f(b_1) - b_1 . f(a_1)}{f(b_1) - f(a_1)}$$

- If $f(p_1) . f(a_1) < 0$ $\quad a_2 = a_1 \quad b_2 = p_1$
- If $f(p_1) . f(b_1) < 0$ $\quad a_1 = p_1 \quad b_1 = b_2$

Examples

4-) $f(x) = 3x + \sin x - e^x \quad x \in [0,1]$

Exact answer $= 0{,}36042170 2 ..$

BISECTION

Step	a	b	p	f(a)	f(b)	f(p)
1	0	1	0,5	−1	1.1232	0,3307
2	0	0,5	0,25	−1	0.3307	−0,2866
3	0,25	0,5	0,375	−0,2866	0,3307	0,0363
4	0,25	0,375	0,3125	−0,2866	0,0363	−0,1219
5	0,3125	0,375	0,3438	−0,1219	0,0363	−0,042

$E_x = |0{,}3604217029 6 - 0{,}3438| = 0{,}0167$

FALSE POSITION

Step	a	b	p	f(a)	f(b)	f(p)
1	0	1	0,4710	−1	1.1232	0,2652
2	0	0,4710	0,3723	−1	0,2652	0,0295
3	0	0,3723	0,3616	−1	0,0295	0,0029
4	0	0,3616	0,3605	−1	0,0029	0,0003
5	0	0,3605	0,3604	−1	0,0003	0,0000

$E_x = 1{,}17 \times 10^{-5}$

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5-) Let $f(x) = -x^3 - \cos x$ on $[-1,0]$ find p_3 using the False Position Method.

Step	a	b	p	f(a)	f(b)	f(p)
1	-1	0	-0,6851	0,4597	-1	-0,4529
2	-1	-0,6851	-0,8414	0,4597	-0,4529	-0,0709
3	-1	-0,8414	-0,8625	0,4597	-0,0709	-0,0088

6-) Consider the equation

$$x^3 - 2x^2 - 5 = 0$$

which has a root $p = 2,690647448028 78\ldots$ on the interval $[2,3]$

a) Bisection 10^{-2}

b) False Position 10^{-2}

c) which method was better?

BISECTION

Step	a	b	p	f(a)	f(b)	f(p)
1	2	3	2,5	-5	4	-1,875
2	2,5	3	2,75	-1.875	4	0,671875
3	2,5	2,75	2,625	-1.875	0,681877	0,6933
4	2,625	2,75	2,6875	-0,6933	0,671876	0,0342

FALSE POSITION

Step	a	b	p	f(a)	f(b)	f(p)
1	2	3	2,5555	-5	4	-1,3717
2	2,5555	3	2,669	-1,3717	4	-0,234272
3	2,669	3	2,6873	-0,234272	4	-0,026
X						

Newton – Raphson Method

$$X_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Algorithm

1-) Choose x_0

2-) Calculate $f(x_i)$ and $f'(x_i)$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$f'(x_i)=0 \rightarrow$ Newton failed

$$\left.\begin{array}{l} \text{If } |x_{i+1} - x_i| < Tol \\ \dfrac{|x_{i+1} - x_i|}{|x_{i+1}|} < Tol \\ f(x_{i+1}) < Tol \end{array}\right\} \text{Stop}$$

3-) Output x_{i+1}

Examples

7-) Use Newton Method to find a root of the equation: Exact solution $= 0,36042170296$ $\quad Tol = 10^{-6}$

$3x + \sin x - e^x = 0 \quad [0,1]$

$f(x) = 3x + \sin x - e^x$

$f'(x) = 3 + \cos x - e^x$

$$\left.\begin{array}{l} f(0) = -1 \\ f'(0) = 3 \end{array}\right\} x_1 = 0 - \frac{-1}{3} = \frac{1}{3}$$

$$\left.\begin{array}{l} f\left(\frac{1}{3}\right) = 1 + 0,32719 - 1,395612 = 0,0684224 \\ f'\left(\frac{1}{3}\right) = 3 + 0,94495 - 1,395612 = 2.549338 \end{array}\right\} x_2 = \frac{1}{3} - \frac{-0,0684224}{2.549338} = 0,3601726$$

$$\left.\begin{array}{l} f(0,3601726) = -6,29 \times 10^{-4} \\ f'(0,3601726) = 2.502253 \end{array}\right\} x_3 = 0,3601726 - \frac{-6,29 \times 10^{-4}}{2.502253} = 0,36421680$$

8-) Use Newton Rapson Method with initial guess $P_0=2$ to approximate root of the equation

$$x^3-2x^2-5=0$$

in the interval $[2,3]$ within an accuracy 10^{-4} $p=2{,}69064744802878$

$f(x)=x^3-2x^2-5$

$f'(x)=3x^2-4x$

$\left.\begin{array}{l} f(2)=-5 \\ f'(2)=4 \end{array}\right\} x_1=2-\left(\frac{-5}{4}\right)=3{,}25$ $\quad E_x=0{,}559 > Tol$

$\left.\begin{array}{l} f(3{,}25)=8.203125 \\ f'(3{,}25)=18.6875 \end{array}\right\} x_2=3{,}25-\left(\frac{8{,}203125}{18{,}6875}\right)=2{,}811037$ $\quad E_x=0{,}1204 > Tol$

$\left.\begin{array}{l} f(2{,}811037)=1.40875 \\ f'(2{,}811037)=12.46164 \end{array}\right\} x_3=2{,}811037-\left(\frac{1.40875}{12.46164}\right)=2{,}69799$ $\quad E_x=0{,}00734 > Tol$

$\left.\begin{array}{l} f(2{,}69799)=0{,}08077 \\ f'(2{,}69799)=11{,}0455 \end{array}\right\} x_4=2{,}69799-\left(\frac{0{,}08077}{11{,}0455}\right)=\underline{2{,}690677}$ $\quad E_x=0{,}000029 < Tol$