VARYANS eğitim

MAT3012-MIDTERM 2.DERS

VECTOR and MATRICES

- Row vector $= [2\ 4\ 5]$
- Column vector $= \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$
- $A = a_{ij_{m\times n}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$

  column   row

- Transpose of Matrix

  A of size $m\times n$ is the matrix of A^T of size $n\times m$

- $(A^T)^T = A$
- $(A.B)^T = A^T.B^T$
- The addition matrix

  $A = \begin{bmatrix} 1 & 2 \\ -2 & 5 \end{bmatrix}, B = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix} \Rightarrow A+B = \begin{bmatrix} 4 & 2 \\ -1 & 7 \end{bmatrix}$

- The product of matrices $A_{(m\times n)}$ by $B_{(n\times p)} = C_{(m\times p)}$

  $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} \rightarrow C = \begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix}$

  $a = 1\times 1 + 2\times -1 = -1$

  $b = 3\times 1 + 4\times -1 = -1$

  $c = 1\times 0 + 2\times 1 = 2$

  $d = 3\times 0 + 4\times 1 = 4$

  $e = 1\times 1 + 2\times 0 = 1$

  $f = 3\times 1 + 4\times 0 = 3$

  $\Big\} C = \begin{bmatrix} -1 & 2 & 1 \\ -1 & 4 & 3 \end{bmatrix}$

- Determinant

  $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\times a_{22} - a_{12}\times a_{21}$

  $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\cdot \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

3-) Gauss Elimination Method

- First suppose we have a system of equation which is on upper triangular form
- And backward substitution

Example: $\begin{cases} x_1 + x_2 - 2x_3 = -3 \\ 3x_2 + 4x_3 = -1 \\ 5x_3 = 10 \end{cases}$ $\quad x_3 = 2$, $x_2 = -3$, $x_1 = 4$

Important !!!

- The order of two rows can be changed
- Multiplying a row by nonzero constant
- The row can be replaced by the sum of that row and nonzero multiple of any other row

Example

$4x_1 - 2x_2 + x_3 = 15$
$-3x_1 - x_2 + 4x_3 = 8$
$x_1 - x_2 + 3x_3 = 13$

$\Rightarrow \left[\begin{array}{ccc|c} 4 & -2 & 1 & 15 \\ -3 & -1 & 4 & 8 \\ 1 & -1 & 3 & 13 \end{array}\right]$ $\begin{array}{l} R_2 = 3R_1 + 4R_2 \\ R_3 = R_1 - 4R_3 \end{array}$ $\Rightarrow \left[\begin{array}{ccc|c} 4 & -2 & 1 & 15 \\ 0 & -10 & 19 & 77 \\ 0 & 2 & -11 & -37 \end{array}\right]$ $R_3 = R_2 + 5R_3$

$\left[\begin{array}{ccc|c} 4 & -2 & 1 & 15 \\ 0 & -10 & 19 & 77 \\ 0 & 0 & -36 & -108 \end{array}\right]$

$x_3 = \frac{108}{36} = \boxed{3}$

$-10x_2 + 19x_3 = 77$
$-10x_2 + 57 = 77$
$-20 = 10x_2$
$\boxed{x_2 = -2}$

$4x_1 - 2x_2 + x_3 = 15$
$4x_1 + 4 + 3 = 15$
$4x_1 = 8$
$\boxed{x_1 = 2}$

Gauss Elimination with Partial Pivoting

Example

$\begin{cases} x_1 + 5x_2 + 3x_3 = 10 \\ x_1 + 3x_2 + 2x_3 = 5 \\ 2x_1 + 4x_2 - 6x_3 = -4 \end{cases}$

$\left[\begin{array}{ccc|c} 1 & 5 & 3 & 10 \\ 1 & 3 & 2 & 5 \\ 2 & 4 & -6 & -4 \end{array}\right] = \left[\begin{array}{ccc|c} 2 & 4 & -6 & -4 \\ 1 & 3 & 2 & 5 \\ 1 & 5 & 3 & 10 \end{array}\right]$ $\begin{array}{l} R_2 = R_2 - \frac{1}{2}R_1 \\ R_3 = R_3 - \frac{1}{2}R_1 \end{array}$ $\Rightarrow \left[\begin{array}{ccc|c} 2 & 4 & -6 & -4 \\ 0 & 1 & 5 & 7 \\ 0 & 3 & 6 & 12 \end{array}\right]$

$\left[\begin{array}{ccc|c} 2 & 4 & -6 & -4 \\ 0 & 3 & 6 & 12 \\ 0 & 1 & 5 & 7 \end{array}\right]$ $R_3 = R_3 - \frac{1}{3}R_2 \Rightarrow \left[\begin{array}{ccc|c} 2 & 4 & -6 & -4 \\ 0 & 3 & 6 & 12 \\ 0 & 0 & 3 & 3 \end{array}\right] \Rightarrow$

$3x_3 = 3$
$\boxed{x_3 = 1}$

$3x_2 + 6x_3 = 12$
$3x_2 + 6 = 12$
$3x_2 = 6$
$\boxed{x_2 = 2}$

$2x_1 + 4x_2 - 6x_3 = -4$
$2x_1 + 8 - 6 = -4$
$2x_1 + 2 = -4$
$2x_1 = -6$
$\boxed{x_1 = -3}$

- Pivoting Method is better.

. Use the Gaussian elimination method with backward substitution and two-digit rounding arithmetic to solve the following linear systems. Do not reorder the equations (The exact solution is $x_1 = 1, x_2 = -1, x_2 = 3$).

$$\begin{cases} 4x_1 - x_2 + x_3 = 8, \\ 2x_1 + 5x_2 + 2x_3 = 3 \\ x_1 + 2x_2 + 4x_3 = 11 \end{cases}$$

$$\left[\begin{array}{ccc|c} 4 & -1 & 1 & 8 \\ 2 & 5 & 2 & 3 \\ 1 & 2 & 4 & 11 \end{array}\right] \begin{array}{l} R_2 = R_2 - \frac{1}{2}R_1 \\ R_3 = R_2 - \frac{1}{4}R_1 \end{array} = \left[\begin{array}{ccc|c} 4 & -1 & 1 & 8 \\ 0 & 5.5 & 1.5 & -1 \\ 0 & 2.25 & 3.75 & 9 \end{array}\right] R_3 = 5{,}5R_3 - 2{,}25R_2$$

$$\left[\begin{array}{ccc|c} 4 & -1 & 1 & 8 \\ 0 & 5{,}5 & 1{,}5 & -1 \\ 0 & 0 & 17.25 & 51.75 \end{array}\right]$$

$17{,}25x_3 = 51{,}75$

$\boxed{x_3 = 3}$

$5{,}5x_2 + 1{,}5x_3 = -1$

$5{,}5x_2 + 4{,}5 = -1$

$5{,}5x_2 = -5{,}5$

$\boxed{x_2 = -1}$

$4x_1 - x_2 + x_3 = 8$

$4x_1 + 1 + 3 = 8$

$4x_1 = 4$

$\boxed{x_1 = 1}$

Use the Gaussian Elimination method to solve the following linear systems. Determine whether trivial pivoting is needed.

a) $$\begin{cases} 2x_1 - 1.5x_2 + 3x_3 = 1, \\ -x_1 \qquad\quad + 2x_3 = 3 \\ 4x_1 - 4.5x_2 + 5x_3 = 1 \end{cases}$$

b) $$\begin{cases} x_1 - x_2 + 3x_3 = 2, \\ 3x_1 - 3x_2 + x_3 = -1 \\ x_1 + x_2 \qquad = 3 \end{cases}$$

a) $$\left[\begin{array}{ccc|c} 2 & -1{,}5 & 3 & 1 \\ -1 & 0 & 2 & 3 \\ 4 & -4{,}5 & 5 & 1 \end{array}\right] \text{pivoting is needed} \Rightarrow \left[\begin{array}{ccc|c} 4 & -4{,}5 & 5 & 1 \\ 2 & -1{,}5 & 3 & 1 \\ -1 & 0 & 2 & 3 \end{array}\right] \begin{array}{l} R_2 = R_2 - \frac{2}{4}R_1 \\ R_3 = 4R_3 + R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 4 & -4{,}5 & 5 & 1 \\ 0 & 0{,}75 & 0{,}5 & 0{,}5 \\ 0 & -4{,}5 & 13 & 13 \end{array}\right] = \left[\begin{array}{ccc|c} 4 & -4{,}5 & 5 & 1 \\ 0 & -4{,}5 & 13 & 13 \\ 0 & 0{,}75 & 0{,}5 & 0{,}5 \end{array}\right] R_3 = 6R_3 + R_2 = \left[\begin{array}{ccc|c} 4 & -4{,}5 & 5 & 1 \\ 0 & -4{,}5 & 13 & 13 \\ 0 & 0 & 16 & 16 \end{array}\right]$$

$16x_3 = 16$

$\boxed{x_3 = 1}$

$-4{,}5x_2 + 13x_3 = 13$

$-4{,}5x_2 + 13 = 13$

$\boxed{x_2 = 0}$

$4x_1 - 4{,}5x_2 + 5x_3 = 1$

$4x_1 + 5 = 1$

$4x_1 = -4$

$\boxed{x_1 = -1}$

Error in Numerical Procedures

Absolute error $= E_x = |x - \hat{x}|$

Relative error $= R_x = \dfrac{|x-\hat{x}|}{|x|}$

- If the numbers are close to 1, $E_x \cong R_x$
- If the numbers are too high, we will use R_x
- If the numbers are too small, we will use E_x

Important Error Definition

- Round-off error: is the error due to storage of finite number of digit.

$p = \frac{22}{7} = 3.1428571428\ldots$

$fl_{chop} = 0.314285 \times 10^{-1}$

$fl_{round} = 0.314286 \times 10^{-1}$

- Input error: is the error in data transfer
- Propagation error: is the error in the output due to error in input data
- Computational error: is the error during arithmetic operations.
- Truncation error: is the error due to the use of approximately expression in place of an exact expression
- Loss of Significance: caused by bad subtraction which means a subtraction of number from another one that is almost equal in value.

Taylor's Theorem

If $f(x)$ has continuous $(n+1)$ derivatives on (a,b) then for any point x and x_0 from (a,b)

$$f(x) = \underbrace{\sum_{k=0}^{n} \frac{f^{k}(x_0)}{k!}\cdot(x-x_0)^k}_{P_n(x)} + \underbrace{\frac{1}{(n+1)!}\cdot f^{n+1}(\varepsilon)\cdot(x-x_0)^{n+1}}_{E_n(x)}$$

Theorem: If $|f^{n+1}(\varepsilon)| \le M$ for any ε between x and x_0 then,

$$|E_n(x)| \le M \cdot \frac{(x-x_0)^{n+1}}{(n+1)!}$$

Example

Using Taylor approximation for the function of $f(x) = e^x$ about $x_0 = 0$ $[0,1]$

The approximation $e^x \cong P_3(x)$ produces truncation error ... bound

$f(x) = e^x = f(0) = 1$

$f'(x) = e^x = f'(0) = 1$

$f''(x) = e^x = f''(0) = 1$

$f'''(x) = e^x = f'''(0) = 1$

$f^{4}(x) = e^x = f^{4}(0) = 1$

$$f(x) = \underbrace{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}}_{P_3(x)} + \underbrace{\frac{f^{4}(\varepsilon)}{4!}\cdot x^4}_{E(x)}$$

$P_3(x) = 1 + 1 + \frac{1}{2} + \frac{1}{6} = \frac{8}{3} = 2{,}6667$

$E(x) = e \ldots = 2{,}78 \cdot \frac{1}{24} = 0{,}1132617$

Let $f(x) = 2x\cos(2x) - (x-2)^2$ and $x_0 = 0$.

a) Find the third Taylor polynomial $P_3(x)$, and use it to approximate $f(0.4)$.

b) Use the error formula in Taylor's Theorem to find an upper bound for the error $|f(0.4) - P_3(0.4)|$. Compute the actual error.

c) Find the third Taylor polynomial $P_4(x)$, and use it to approximate $f(0.4)$.

d) Use the error formula in Taylor's Theorem to find an upper bound for the error $|f(0.4) - P_4(0.4)|$. Compute the actual error.

$f(x) = 2x\cos\cancel{2x} - (x-2)^2$ and $x_0 = 0$

$f'(x) = -4x\sin(2x) + 2\cos(2x) - 2(x-2) = -4x\sin(2x) + 2\cos 2x - 2x + 4$

$f''(x) = -8x\cos(2x) - 8\sin(2x) - 2$

$f'''(x) = 16x\sin(2x) - 24\cos(2x)$

$f^{4}(x) = 32x\cos(2x) + 64\sin(2x)$

$f^{5}(x) = 160\cos(2x) - 64\sin(2x)$

$\Rightarrow f(0) = -4$

$\Rightarrow f'(0) = 6$

$\Rightarrow f''(0) = -2$

$\Rightarrow f'''(0) = -24$

$\Rightarrow f^{4}(0) = 0$

a) $P_3(x) = -4 + 6x - \frac{2x^2}{2} - \frac{24x^3}{6} \Rightarrow -4x^3 - x^2 + 6x - 4$

$f(0{,}4) \cong P_3(0{,}4) = -4(0{,}4)^3 - (0{,}4)^2 + 6.(0{,}4) - 4 =$

$-0{,}256 - 0{,}16 + 2.4 - 4 = -2{,}016$

b) $E_3 = |f(0{,}4) - P_3(0{,}4)|$

$E_3 \leq \left|f^{4}(\varepsilon).\frac{x^4}{4!}\right| \quad \varepsilon \in (0, 0{,}4)$

$f^{4}(\varepsilon) = 32.(0{,}4).\cos(0{,}8) + 64\sin(0{,}8) = 55$

$E_3 = 55.\frac{(0{,}4)^4}{4!} = 0{,}058$

Actual error $= |f(0{,}4) - P_3(0{,}4)|$

$f(0{,}4) = 2(0{,}4).\cos(0{,}8) - (0{,}4-2)^2 = 2{,}002635$

$|2{,}002635 - 2{,}016| = 0{,}013365$

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